International Journal of Theoretical Physics, Vol. 33, No. 9, 1994

Normal Forms of Reversible Dynamical Systems

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Received February 19, 1994

We consider reversible dynamical systems with a fixed point which is also fixed under the reversing involution; we show that applying to such a system the canonical Poincaré–Dulac procedure reducing a dynamical system to its normal form, we obtain a normal form which is still reversible (under the same involution as the original system); conversely, we also show how to obtain all the reversible systems which are reduced to a given reversible form. This allows one to (locally) classify reversible dynamical systems, and reduce their (local) study to that of reversible normal forms.

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

In this note we will deal with *reversible dynamical systems* (Devaney, 1976). Here, by a smooth dynamical system (DS) we mean a system of ODEs

$$\dot{x} = f(x) \tag{1}$$

$$x \in M = \mathbf{R}^m; \qquad f\colon \quad M \to TM$$

where f is a smooth vector field (VF) on M. Since $M = \mathbf{R}^m$, it can also be seen as a smooth function $f\colon \mathbf{R}^m \to \mathbf{R}^m$, and in the following we will consider it in this way. We will consider the case $f(0) = 0$; it is then natural to study locally (1) around the fixed point $x = 0$.

The DS (1) will be called *reversible* if there exists an involution $S\colon M \to M$ such that

$$S^2 = I \tag{2}$$

$$f(Sx) = -Sf(x) \tag{3}$$

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0020-7748/94/0900-1917$07.00/0

The name "reversible" is due to the fact that such a system is invariant under the transformation

$$\begin{cases} x \to Sx \\ t \to -t \end{cases} \tag{4}$$

Reversible systems have been studied for a long time (Devaney, 1976; Sachs, 1987; Moser, 1966, 1967; Sevryuk, 1986; Arnold and Sevryuk, 1986) due to their properties and physical interest; in recent times, they have attracted renewed attention (Quispel *et al.*, 1988, 1989; Quispel and Roberts, 1988, 1989; Quispel and Capel, 1989, 1992; Post *et al.*, 1990; Roberts and Quispel, 1992). We will refer to Devaney (1976), Quispel and Capel (1989), and Roberts and Quispel (1992) and references therein for motivation, properties, and known results concerning reversible systems, and concentrate on the specific problem we want to discuss, namely the connection between reversible dynamical systems and normal forms.

In the study of DS, a valuable tool is given by the theory of (Poincaré) normal forms (NF) (Arnold, 1982; Arnold and Il'yashenko, 1988; Bruno, 1989; Verhulst, 1990); this allows one to classify DS up to differential equivalence, and to extract a set of DS—those in NF—which can be studied in order to obtain information about the most general DS. It should be stressed that other NF theories apply if we are interested in other forms of equivalence, e.g., $\mathscr{C}^k$ or topological ($\mathscr{C}^0$) equivalence rather than the differential ($\mathscr{C}^\infty$) one (Arnold, 1982; Arnold and Il'yashenko, 1988; Bruno, 1989). Here, we will only be concerned with $\mathscr{C}^\infty$ equivalence and Poincaré, or Poincaré–Dulac, NF; from now on, by NF we will mean Poincaré–Dulac NF.

The study of reversible dynamical systems (RDS) encounters many difficulties, one of which is the very fact that, while it is easy to decide if a given DS is reversible with respect to a given involution S, it is not easy at all to know if, given a DS, there exists any S under which this is reversible. It is quite clear that a NF classification of RDS would be quite helpful in this respect (and many others as well). In particular, it would be helpful if we could affirm that a DS which is reversible with respect to (w.r.t.) a given S_0 is reduced to a NF which is still reversible w.r.t. the same S_0.

This is indeed the goal of the present node, and we will prove below that the property mentioned above does actually hold.

The same concept can be expressed in a slightly more abstract way in terms of diagrams, which will offer the opportunity to introduce a notation to be used in the following.

Let $\mathscr{M}$ be the set of smooth VF on M (i.e., of diffeomorphisms of M), which in the present case can be identified with the set of smooth functions $f: M \to M$, and let $\mathscr{R}_s \subset \mathscr{M}$ be the set of VF reversible w.r.t. S, i.e., for which (3) is satisfied, S being such that (2) holds. Let ρ_s be the projection $\rho_s: \mathscr{M} \to \mathscr{R}_s$, and let π be the operator which associates to a VF its canonical NF: let $\mathscr{P}$ be the set of VF in NF, i.e. $\mathscr{P} = \pi(\mathscr{M})$. We would then like to prove that the following diagram is commutative:

$$\begin{array}{ccc} \mathscr{M} & \xrightarrow{\pi} & \mathscr{P} \\ \downarrow{\scriptstyle \rho_s} & & \downarrow{\scriptstyle \rho_s} \\ \mathscr{R}_s & \xrightarrow{\pi} & \mathscr{P}_s \end{array} \tag{5}$$

Notice that *a priori* the sets $\pi(\mathscr{R}_s)$ and $\rho_s(\mathscr{P})$ are not equal. Indeed, $\rho_s(\mathscr{P})$ represents the set NF VF which are in $\mathscr{R}_s$, i.e.,

$$\rho_s(\mathscr{P}) = (\mathscr{P} \cap \mathscr{R}_s) \equiv \mathscr{P}_s \tag{6}$$

It is clear that if $w \in \mathscr{P}_s$, then $w \in \mathscr{R}_s$ and $\pi(w) = w$, so that

$$\mathscr{P}_s \subseteq \pi(\mathscr{R}_s) \equiv \mathscr{P}'_s \tag{7}$$

but it could happen that a reversible VF is mapped by π into a nonreversible NF (or a NF reversible w.r.t. a different S). We will have to prove this is not the case, i.e., $\mathscr{P}'_s = \mathscr{P}_s$.

As already suggested by the notation (2), (3), we will only consider here the case of *linear* involutions: i.e., S will be an $(m \times m)$ real matrix; notice that in this case, $x = 0$ is necessarily a fixed point for S. The case of nonlinear involutions would be considerably more complicated, as the coordinate transformation needed to take the DS to NF would also modify the coordinate representation of S; this is analogous to what happens when considering the NF reduction of symmetric (rather than reversible) DS in the case of nonlinear symmetries, which requires a quite different approach (Cicogna and Gaeta, 1993). We presume that the case of nonlinear involutions can be dealt with along the lines of the nonlinear symmetries approach, but we defer its study to a later time, due to the mentioned substantial difference between the two approaches.

It should also be mentioned that, since as already mentioned in the case $M = \mathbf{R}^m$, f can be seen also as a function $f: \mathbf{R}^m \to \mathbf{R}^m$, our results immediately apply to discrete DS in $\mathbf{R}^m$, i.e., maps of the type $x_{n+1} = f(x_n)$, such that f satisfies (3); anyway, in the case of discrete DS, the appropriate definition of reversibility would not be (3), but instead $f(Sf(x)) = Sx$ (Quispel and Capel, 1989; Roberts and Quispel, 1992).

In the following sections, we will first briefly recall how the NF reduction works (Section 2) and some simple algebraic facts we need (Section 3); we pass then to prove our main result, announced above, in Section 4; in Section 5 we illustrate by a concrete example the simplification introduced by our result in the study of RDS; we summarize and discuss our results in Section 6.

After the completion of the present work, I became aware of the recent paper by Lamb *et al.* in which the authors obtain many relevant results concerning reversible dynamical systems. In particular, the result given in the present note is also contained in their paper (see their Propositions 3 and 4), although it is obtained there by a different method. I think it is of some interest to see this result also by the (simpler) method presented here, which is elementary and completely explicit.

2. REDUCTION TO POINCARÉ–DULAC NORMAL FORM

The Poincaré–Dulac procedure (Arnold, 1982) to transform a DS (equivalently, a VF) like (1) in its NF, given by

$$\dot{x} = w(x); \qquad w = \pi(f) \tag{8}$$

is based on a sequel of formal near-identity transformations of the form

$$x \to \tilde{x} = x + h_k(x), \qquad k \geq 2 \tag{9}$$

$$h_k(ax) = a^k h_k(x$$

Under this, and with ∇h the matrix function given by

$$(\nabla h)^i_j = \frac{\partial h^i}{\partial x^j} \tag{10}$$

the DS (1) is transformed into

$$\dot{x} = [I + (\nabla h)]^{-1} f(x + h_k(x)) \equiv \tilde{f}(x) \tag{11}$$

If now we assume $f(0) = 0$, write f as a sum of homogeneous terms

$$f(x) = \sum_n f_n(x) \equiv Ax + \sum_{n=2}^{\infty} f_n(x) \tag{12}$$

and expand (11) up to terms of order k, we get

$$\dot{x} = Ax + \sum_{n=2}^{k-1} f_n(x) + [f_k(x) + Ah_k(x) - (\nabla h)Ax] + \text{h.o.t.} \tag{13}$$

where h.o.t. represents terms homogeneous of order $(k + 1)$ and higher. In other terms, under (9) the terms of order $n < k$ are unchanged, and that of order k is changed according to

$$f_k \to \tilde{f}_k = f_k - \mathscr{L}_A(h_k) \tag{14}$$

where $\mathcal{L}_A : \mathcal{M} \to \mathcal{M}$ is the *homological operator* associated to A,

$$\mathcal{L}_A = Ax \cdot \nabla - A \tag{15}$$

also equivalent to the Poisson bracket with the linear field Ax; i.e., $\forall f \in \mathcal{M}$ it satisfies

$$(\mathcal{L}_A f)(x) = \{Ax, f(x)\} \equiv (Ax \cdot \nabla) f(x) - (f(x) \cdot \nabla) Ax \tag{16}$$

Let us introduce the projection θ on the range of $\mathcal{L}_A$,

$$\theta: \quad \mathcal{M} \to \mathrm{Ran}(\mathcal{L}_A) \tag{17}$$

Then, the *homological equation*

$$\mathcal{L}_A h_k = \theta(f_k); \qquad k \geq 2 \tag{18}$$

can be uniquely solved for h_k up to a $\delta h_k \in \mathrm{Ker}(\mathcal{L}_A)$. By choosing the h_k which solves (18) as the h_k generating (9), and proceeding sequentially for $k = 2, 3, \ldots,$ we can then eliminate from f the terms in $\mathrm{Ran}(\mathcal{L}_A)$ up to any desired order. The NF will therefore be characterized by the fact that it includes only terms in $[\mathrm{Ran}(\mathcal{L}_A)]^c$, i.e., in $\mathrm{Ker}(\mathcal{L}_A^+)$.

It is customary in NF theory to restrict to the case A is a normal matrix, i.e.,

$$[A, A^+] = 0 \tag{19}$$

in which case also $[\mathcal{L}_A, \mathcal{L}_A^+] = 0$, and we have in particular

$$\mathcal{M} = \mathrm{Ker}(\mathcal{L}_A) \oplus \mathrm{Ran}(\mathcal{L}_A) \tag{20}$$

In the following we will assume (19) and (20) hold (Assumption A). The NF reduction in the case A has Jordan blocks is discussed, e.g., in (Arnold, 1982; Arnold and Il'yashenko, 1988; Bruno, 1989); it would be easy to extend our results to this case (see the final discussion).

As mentioned above, the solution to (18) is unique provided we require $h_k \in [\mathrm{Ker}(\mathcal{L}_A)]^c$; if Assumption A is satisfied, this amounts to requiring $h_k \in \mathrm{Ran}(\mathcal{L}_A)$. We speak then of *canonical* NF reduction.

3. SOME ALGEBRAIC FACTS

Let us consider a matrix S satisfying (2). To this matrix we will associate two sets of VF, i.e., the set $\mathcal{C}_s \subset \mathcal{M}$ of VF commuting with S, and the set $\mathcal{R}_s$ of VF, which are reversible w.r.t. S (anticommuting with S):

$$\mathcal{C}_s = \{f \in \mathcal{M} \,/\, f(Sx) = Sf(x)\} \subset \mathcal{M} \tag{21}$$

$$\mathcal{R}_s = \{f \in \mathcal{M} \,/\, f(Sx) = -Sf(x)\} \subset \mathcal{M} \tag{22}$$

Because of (2), these give a decomposition of $\mathcal{M}$, i.e., we have the following result:

Lemma I′. If $S^2 = I$, then $\mathcal{M} = C_s \oplus \mathcal{R}_s$.

The above is equivalent to affirming the following.

Lemma I″. If $S^2 = I$, for any $f \in \mathcal{M}$ there are $\phi \in \mathcal{C}_s$, $\psi \in \mathcal{R}_s$ such that $f(x) = \phi(x) + \psi(x)$.

Proof. This follows from standard general group-theoretic results (Kirillov, 1976; Hamermesh, 1962), it is also easy to give a direct proof. Using the decomposition of the lemma, we get

$$f(Sx) = \phi(Sx) + \psi(Sx) = S\phi(x) - S\psi(x) = Sf(x) - 2S\psi(x) \tag{23}$$

so that (recall $S^{-1} = S$) it suffices to define

$$\psi(x) = -\frac{1}{2}[Sf(Sx) - f(x)] \tag{24}$$

to have the desired decomposition. ■

Let us now consider $h \in \mathcal{C}_s$ (or $h \in \mathcal{R}_s$), and the associated matrix $(\nabla h)(x) \equiv \nabla_j h^i(x)$. We have the following result:

Lemma II′. If $h \in \mathcal{C}_s$, then $(\nabla h)(Sx) = S[(\nabla h(x)]S^{-1}$.

Lemma II″. If $h \in \mathcal{R}_s$, then $(\nabla h)(Sx) = -S[(\nabla h)(x)]S^{-1}$.

Proof. This is again a well-known fact, but we give a short proof here. Consider the function

$$\tilde{h}(x) \equiv h(Sx) \tag{25}$$

Its gradient is

$$(\nabla\tilde{h})(x) = (\nabla h)(Sx) \cdot S \tag{26}$$

but using $h(Sx) = \pm Sh(x)$, we also have $\tilde{h} = \pm Sh$, i.e.,

$$(\nabla\tilde{h})(x) = \pm S \cdot (\nabla h)(x) \tag{27}$$

and therefore the lemma. ■

4. PROOF OF THE MAIN RESULT

First of all, we notice that, since S is a linear operator, (3) also implies

$$f_n(Sx) = -Sf_n(x) \tag{28}$$

which enforces in particular for the constant term $f_0(x) \equiv f_0$ that $Sf_0 = -f_0$; we will *assume* that actually

$$f_0(x) \equiv 0 \tag{29}$$

so that the r.h.s. decomposition of (12) applies. Moreover, for $n = 1$ we get

$$AS = -SA \tag{30}$$

We can now proceed to prove the main theorem; we divide the proof into a few steps, and assume (2) and (3) are satisfied.

Lemma III. If A satisfies (30), then $\mathscr{L}_A$ exchanges $\mathscr{C}_s$ and $\mathscr{R}_s$, i.e., $\mathscr{L}_A: \mathscr{C}_s \to \mathscr{R}_s$ and $\mathscr{R}_s \to \mathscr{C}_s$.

Proof. This follows by direct computation using Lemma II. Indeed, let us write explicitly

$$(\mathscr{L}_A h)(x) \equiv g(x) = [(\nabla h(x)] A_x - Ah(x) \tag{31}$$

Now, consider

$$g(Sx) = [(\nabla h)(Sx)] ASx - Ah(Sx) \tag{32}$$

which for $h(Sx) = \pm Sh(x)$ gives, due to Lemma II and to (30),

$$\begin{aligned} g(Sx) &= \pm S[(\nabla h)(x)] S^{-1} ASx \mp ASh(x) \\ &= \mp [(\nabla h)(x)] Ax \pm SAh(x) = \mp Sg(x) \end{aligned} \tag{33}$$

and the lemma is proved. ■

Corollary. If $\mathscr{L}_A h \in \mathscr{R}_s$, then $h = h_0 + \delta h$, with $h_0 \in \mathscr{C}_s$ and $\delta h \in \mathrm{Ker}(\mathscr{L}_A)$.

Corollary. If $g \in [\mathscr{R}_s \cap \mathrm{Ran}(\mathscr{L}_A)]$, then $\mathscr{L}_A h = g$ admits a solution $h \in \mathscr{C}_s$.

It should be stressed that $\mathscr{L}_A$ controls the transformation of the term f_k of the same order as the function h_k generating the transformation (9), but *not* the transformation induced in higher-order terms. In particular, it could happen *a priori* that the action of (9) with $h_k \in \mathscr{C}_s$ transforms a reversible VF into a nonreversible one. This eventuality is excluded by the following result.

Lemma IV. If $f \in \mathscr{R}_s$ and $h \in \mathscr{C}_s$, then f is transformed by (9) into a new reversible VF $\tilde{f} \in \mathscr{R}_s$.

Proof. The explicit expression of $\tilde{f}$ is given in (11); by this we can check that indeed, using Lemma II and $h \in \mathscr{C}_s$,

$$\begin{aligned} \tilde{f}(Sx) &= [I + (\nabla h)(Sx)]^{-1} f(Sx + h(Sx)) \\ &= [SIS^{-1} + S \cdot (\nabla h)(Sx) \cdot S^{-1}]^{-1} f(Sx + Sh(x)) \\ &= -S \cdot [I + (\nabla h)(Sx)]^{-1} \cdot S^{-1} Sf(x + h(x)) \\ &\equiv -S\tilde{f}(x) \end{aligned}$$

and the lemma is proved. ■

It should be stressed that in this section neither the condition (2) nor Assumption A has been used up to now; we use them in the following result.

Lemma V. Let $S^2 = I$ and let A satisfy Assumption A; let θ be as in (17). Then if $f \in \mathscr{R}_s$, also $\theta f \in \mathscr{R}_s$.

Proof. Since, by assumption, (2) and (19) are verified, we are granted that Lemma I and the decomposition (20) hold. We can therefore decompose any function $f \in \mathscr{M}$ as

$$f(x) = \phi_+(x) + \phi_-(x) + \psi_+(x) + \psi_-(x) \tag{34}$$

where $\phi_\pm \in \mathrm{Ker}(\mathscr{L}_A)$, $\psi_\pm \in \mathrm{Ran}(\mathscr{L}_A)$, and

$$\chi_\pm(Sx) = \pm S\chi(x), \qquad \chi = \phi, \psi \tag{35}$$

Since $f \in \mathscr{R}_s$, we must have

$$\phi_+(x) = -\psi_+(x) \tag{36}$$

and since they belong to complementary (due to Assumption A) linear spaces, necessarily

$$\phi_+ = \psi_+ = 0 \tag{37}$$

which enforces in particular

$$\theta(f) = \psi_- \in \mathscr{R}_s \tag{38}$$

and proves the lemma. ■

This also concludes the proof of the ingredients we need for the main result, which we state as follows.

Theorem. Let $S^2 = I$ and $f \in \mathscr{R}_s$; let $A = (Df)(0)$ satisfy Assumption A, i.e., $[A, A^+] = 0$. Then the canonical reduction of f to NF is reversible w.r.t. the same S as f, i.e., $w \equiv \pi(f) \in \mathscr{R}_s$.

Corollary. $\pi(\mathscr{R}_s) \equiv \mathscr{P}'_s = \mathscr{P}_s \equiv \mathscr{P} \cap \mathscr{R}_s$.

Proof. It is clear that, by Lemma IV, if the NF transformation is operated by a sequel of coordinate transformations (9) with all generators $h_k \in \mathscr{C}_s$, then $\pi: \mathscr{R}_s \to \mathscr{R}_s$. The h_k are determined as solutions to the homological equation (18); by Lemma III, if the r.h.s. of this is in $\mathscr{R}_s$ and (30) holds, it is indeed possible to choose $h_k \in \mathscr{C}_s$, and this corresponds to the canonical choice. The fact that (30) holds is guaranteed by $f \in \mathscr{R}_s$ (see above), and $\theta(f_k) \in \mathscr{R}_s$ follows from Lemma V. The proof of the theorem is complete. The proof of the corollary is immediate, since the theorem shows that

$$\pi(\mathscr{R}_s) \equiv \mathscr{P}'_s \subseteq \mathscr{P} \cap \mathscr{R}_s \equiv \mathscr{P}_s \tag{39}$$

On the other hand, as observed in Section 1, since π is the identity on $\mathscr{P}$, so that $\pi(\mathscr{P} \cap \mathscr{R}_s) = \mathscr{P} \cap \mathscr{R}_s$, necessarily $\mathscr{P}_s \subseteq \mathscr{P}'_s$, so that $\mathscr{P}_s = \mathscr{P}'_s$. ■

It should be stressed that our proof also permits one to identify the RDS which correspond to a given reversible NF: indeed, from the proof it is clear that we have the following.

Corollary. Given a NF VF $w(x) \in \mathscr{P}_s$, the most general RDS f such that $\pi(f) = w$ can be obtained by unfolding the NF w by means of coordinate transformations of the type (9) with $h_k \in \mathscr{C}_s$.

5. EXAMPLE

We would like to illustrate briefly by a concrete example the applications of our result. We will fix A and S, given by

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \qquad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{40}$$

Notice that $A = A^+$ and $\mathscr{L}_A^+ = \mathscr{L}_A$, i.e., Assumption A is verified.

We will write

$$f(x, y) = \begin{pmatrix} f^+(x, y) \\ f^-(x, y) \end{pmatrix} \tag{41}$$

By straightforward algebra, we get that

$$f \in \mathscr{R}_s \leftrightarrow \begin{cases} f^+(x, y) = \sum_{l,m} \alpha_{l,m} x^l y^m \\ f^-(x, y) = \sum_{l,m} -\alpha_{l,m} x^m y^l \end{cases} \tag{42}$$

$$f \in \operatorname{Ker}(\mathscr{L}_A) \leftrightarrow \begin{cases} f^+(x, y) = \sum_m \beta_m (xy)^m x \\ f^-(x, y) = \sum_m \gamma_m (xy)^m y \end{cases} \tag{43}$$

If we want to study all the DS which are reversible w.r.t. S and whose linear part is given by A, we should in principle study a generic DS given by (42) with the only constraint $\alpha_{1,0} = 1$, $\alpha_{0,1} = 0$.

The theory of NF tells us that given any system f with linear part A, we can reduce the problem to that of studying a system given by $w = \pi(f) \in \operatorname{Ker}(\mathscr{L}_A)$, i.e., a system of the form (43) with $\beta_0 = -\gamma_0 = 1$.

The theorem proved above ensures that we can consider only reversible NF, i.e., we only have to study $f \in (\mathscr{R}_s \cap \operatorname{Ker}(\mathscr{L}_A))$; in the present case [cf. (42), (43)] we have

$$f \in (\mathscr{R}_s \cap \operatorname{Ker}(\mathscr{L}_A)) \leftrightarrow \begin{cases} f^+(x, y) = \sum_m \beta_m (xy)^m x \\ f^-(x, y) = \sum_m -\beta_m (xy)^m y \end{cases} \tag{44}$$

It is clear that this represents a great simplification with respect to either (42) and (43).

To illustrate this in a completely explicit case, consider A, S as in (40), and say we are interested only in DS up to cubic terms. From (42) we have that the general form of such a system is

$$\begin{aligned} \dot{x} &= x + c_1 x^2 + c_2 xy + c_3 y^3 + c_4 x^3 + c_5 x^2 y + c_6 xy^2 + c_7 y^3 \\ \dot{y} &= -y - c_3 x^2 - c_2 xy - c_1 y^2 - c_7 x^3 - c_6 x^2 y - c_5 xy^2 - c_4 y^3 \end{aligned} \tag{45}$$

while with our theorem it suffices to consider

$$\begin{aligned} \dot{x} &= x + ax^2 y \\ \dot{y} &= -y - axy^2 \end{aligned} \tag{46}$$

It is also easy to check explicitly that the transformation (9) taking (45) into (46) is generated by $h_2, h_3 \in \mathscr{C}_s$. Indeed we have explicitly

$$h_2 = \begin{pmatrix} h_2^+ \\ h_2^- \end{pmatrix} = \begin{pmatrix} c_1 x^2 - c_2 xy - (1/3) c_3 y^2 \\ c_1 y^2 - c_2 xy - (1/3) c_3 x^2 \end{pmatrix} \tag{47}$$

$$h_3 = \begin{pmatrix} h_3^+ \\ h_3^- \end{pmatrix} = \frac{1}{2} \begin{pmatrix} c_4 x^3 - c_6 xy^2 - (1/2) c_7 y^3 \\ c_4 y^3 - c_6 x^2 y - (1/2) c_7 x^3 \end{pmatrix} \tag{48}$$

6. DISCUSSION AND CONCLUSIONS

In order to obtain our results, we have introduced two restrictive assumptions, i.e., Assumption A on one hand, and $M = \mathbf{R}^m$ on the other. We would now like to discuss briefly how our discussion is affected if these assumptions are relaxed.

Assumption A, i.e., (19), was used in Lemma V, where we employed the decomposition (20) for $\mathscr{M}$. If (19) fails, we can still decompose $\mathscr{M}$ in terms of $\operatorname{Ran}(\mathscr{L}_A)$ and $\operatorname{Ker}(\mathscr{L}_A^+)$. We should consider $\phi_\pm \in \operatorname{Ker}(\mathscr{L}_A^+)$; (34) and (36) would still hold, and the latter would again imply (37) and therefore (38).

The reason to consider Assumption A is that the reduction to NF is slightly more complicated if Assumption A is not satisfied; it would still be possible to go along with the lines of the present discussion and obtain the same final result. The reduction to NF when Assumption A does not hold is discussed in Arnold (1982), Arnold and Il'yashenko (1988), and Bruno (1989) in the general case and in Belitsky (1979) and Elphick *et al.* (1987) for the symmetric case (39); see below.

As for the assumption $M = \mathbf{R}^m$, if we consider instead a generic smooth manifold embedded in $\mathbf{R}^m$, $M \subset \mathbf{R}^m$, then the very definition (3) of RDS we used here is not adequate: indeed, if $f(x) \in T_x M$, $f(Sx) \in T_{Sx} M$, and we should make use of a connection on TM in order to express the relation between tangent VF at different points of M. For curved manifolds, the value of the transported VF would depend on the path used to go from x to Sx, so that (3) would not make sense in general. It would still make sense if we assume that x_0 is a fixed point both for S and for the VF f, as we have assumed in our discussion (obviously we can take any point $x_0 \in M$ as $x = 0$).

More precisely, in general we should revert to local definitions of reversibility (Quispel and Capel, 1989); in this case, we would require that there are a point x_0 which is a fixed point for S, i.e., $Sx_0 = x_0$, and a disk $\mathscr{B}$ around x_0 (in the matrix on M) invariant under S. If M is μ-dimensional, we can then project the disk $\mathscr{B}$ on the tangent hyperplane to M in x_0, i.e., to a space $\mathbf{R}^\mu$, provided the radius of $\mathscr{B}$ is small enough. We are then in the setting of our discussion, with the role of $\mathscr{M}$ taken by $\mathscr{B}$ and that of m taken by μ. Our discussion and results therefore apply to a local NF classification of RDS in the neighborhood of a common fixed point of S and f. These considerations do also extend to the case of maps, as mentioned in Section 1.

Summarizing, we have proved that the (canonical) Poincaré–Dulac NF reduction of a DS reversible w.r.t. a given linear involution S is still reversible w.r.t. the same S.

Our theorem allows us to reduce the local study of RDS to that of reversible NF, in the same way as one can reduce the local study of DS to that of NF.

It should be mentioned that the result obtained here is the equivalent of the known one for DS which are symmetric under a linear transformation, i.e., such that

$$f(Sx) = Sf(x) \tag{49}$$

holds, rather than (3). In this case, indeed, we are granted that the NF $w = \pi(f)$ satisfies as well $w(Sx) = Sw(x)$ (Arnold and Il'yashenko, 1988; Belitsky, 1979; Elphick *et al.*, 1987). This result cannot be directly applied to reversible systems, as the latter invariant under the transformation (4), which acts not only on the x, but on the time t as well.

In the case (39) the equivariant NF theory has been recently generalized to consider nonlinear transformations S (Cicogna and Gaeta, 1993); we conjecture that the parallel between the reversible and the equivariant cases extends to this setting. Notice, however, that if the linear part of S is nonresonant, it is possible by the Poincaré–Dulac procedure to linearize S and deal with the case of linear symmetries considered in the present work.

REFERENCES

Arnold, V. I. (1982). *Geometrical Methods in the Theory of Differential Equations*, Springer, Berlin.

Arnold, V. I., and Il'yashenko, Yu. S. (1988). Ordinary differential equations, in *Encyclopaedia of Mathematical Sciences — Vol. I (Dynamical Systems I)*, D. V. Anosov and V. I. Arnold, eds., Springer, Berlin.

Arnold, V. I., and Sevryuk, M. B. (1986). Oscillations and bifurcations in reversible systems, in *Nonlinear Phenomena in Plasma Physics and Hydrodynamics*, R. Z. Sagdeev, ed., Mir, Moscow.

Belitsky, G. R. (1979). *Normal Forms, Invariants, and Local Mappings*, Naukova Dumka, Kiev [in Russian].

Bruno, A. D. (1989). *Local Methods in Nonlinear Differential Equations*, Springer, Berlin.

Cicogna, C., and Gaeta, G. (1993). Poincaré normal forms and Lie-point symmetries, preprint.

Devaney, R. L. (1976). *Transactions of the American Mathematical Society*, **218**, 89.

Elphick, C., Tirapegui, E., Brachet, M. E., Coullet, P., and Ioos, G. (1987). *Physica D*, **29**, 95.

Hamermesh, M. (1962). *Group Theory*, Addison-Wesley, Reading, Massachusetts [reprinted Dover, New York, 1991].

Kirillov, A. A. (1976). *Elements of the Theory of Representations*, Springer, Berlin.

Moser, J. (1966). *SIAM Review*, **8**, 145.

Moser, J. (1967). *Mathematische Annalen*, **169**, 136.

Post, T., Capel, H. W., Quispel, G. R. W., and van der Weele, J. P. (1990). *Physica A*, **164**, 625.

Quispel, G. R. W., and Capel, H. W. (1989). *Physics Letters A*, **142**, 112.

Quispel, G. R. W., and Capel, H. W. (1992). *Physics Letters A*, **162**, 243.

Quispel, G. R. W., and Roberts, J. A. G. (1988). *Physics Letters A*, **132**, 161.

Quispel, G. R. W., and Roberts, J. A. G. (1989). *Physics Letters A*, **135**, 337.

Quispel, G. R. W., Roberts, J. A. G., and Thompson, C. J. (1988). *Physics Letters A*, **126**, 419.

Quispel, G. R. W., Roberts, J. A. G., and Thompson, C. J. (1989). *Physica D*, **34**, 183.

Roberts, J. A. G., and Quispel, G. R. W. (1992). *Physics Reports*, **216**, 63.

Sachs, R. G. (1987). *The Physics of Time Reversal*, University of Chicago Press, Chicago.

Sevryuk, M. B. (1986). *Reversible Systems*, Springer, Berlin.

Verhulst, F. (1990). *Nonlinear Differential Equations and Dynamical Systems*, Springer, Berlin.